Consider a two-level system where the kets $|\psi_1>$ and $|\psi_2>$ form an orthonormal basis. Now define a new basis $|\phi_1>$ and $|\phi_2>$ by

$|\phi_1> = \frac{1}{\sqrt{2}}(|\psi_1> + |\psi_2>)$ and

$|\phi_2> = \frac{1}{\sqrt{2}}(|\psi_1> - |\psi_2>)$

An operator $\hat{O}$ represented in $|\psi_i>$ basis is given by

$\hat{O} = \begin{bmatrix} 1 & a \\ a & 1 \end{bmatrix}$

Find the representation of $\hat{O}$ in the new basis. Here 'a' is a constant.
2. Consider a one-dimensional system described by the Hamiltonian $H = \frac{p^2}{2m} + V(x)$.

(a) Show that $[H, x] = -i\hbar \frac{\partial}{\partial x}$.

(b) Find the expectation value of $\hat{P}$ in an eigen state of the given $H$.

Hint: Use the result obtained in (a) to evaluate (b).

3. A particle of mass $m$ is confined within an infinite 1-D well, between $x = 0$ and $x = L$. The eigen states of the Hamiltonian are given by

$$\langle x | \phi_n \rangle = \sqrt{\frac{2}{L}} \sin \left( \frac{n\pi x}{L} \right)$$

where $E_n = \frac{n^2\hbar^2}{2mL}$, $n = 1, 2, \ldots$.

Consider a situation where at time $t = 0$ the particle is in state $|\psi(t=0)\rangle = \frac{|\phi_1\rangle + |\phi_2\rangle}{\sqrt{2}}$.

(a) Find the time-dependent $|\psi(t)\rangle$.

(b) Calculate the coordinate representation of $|\psi(t)\rangle$.

(c) Check if $|\psi(t)\rangle$ comes back to $|\psi(t=0)\rangle$ after a certain time $t_0$. If so determine $t_0$.  

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4. Using the uncertainty relation $\Delta x \Delta p \geq \hbar / 2$, estimate the ground state energy of 1-D harmonic oscillator.

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2} \hat{x}^2$$

Given $\langle x \rangle = \langle p \rangle = 0$ for all eigen state of $\hat{H}$ and $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$ and $\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$. 

Hint: Express expectation value of $\hat{H}$ in terms of $\Delta x$ and $\Delta p$ and then use uncertainty relation and minimization condition.

5. In the simultaneous eigen basis of $\hat{L}_z$ and $\hat{L}_z$ which is represented as $|l, m\rangle$, obtain the matrix representation of $L_x$ and $L_y$ operator for $l = 1$ case.

Given, $L_+ = L_+ + i L_y$

$L_- = L_+ - i L_y$

$L_{z \uparrow} |l, m\rangle = \hbar \sqrt{l(l+1) - m(m+1)} |l, m \pm 1\rangle$

$L_{z \downarrow} |l, m\rangle = m \hbar |m\rangle$

$L^2 |l, m\rangle = l(l+1) \hbar^2 |m\rangle$

And the basis is given by $\{|l L, m = 0, \pm 1\rangle\}$

6. (a) Show that, in 1st order time independent perturbation theory, the 1st order correction to the $m^{th}$ eigen state is orthogonal to the eigen state itself.

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(b) Consider a particle of charge \( q \) and mass \( m \) in a harmonic oscillator potential \( V = \frac{1}{2} mw^2 \lambda^2 \). Now apply a small electric field of magnitude \( f' \) along the +ve \( \lambda \)-axis.

So, \( H = H_0 + H' \)

\[
\frac{\hat{P}^2}{2m} + \frac{1}{2} mw^2 \lambda^2 + cf' \lambda
\]

Treating the electric field perturbatively, calculate the energy correction to \( n^{th} \) eigen state in 1st order perturbation theory.

Given:

\[
a = \sqrt{\frac{mw}{2\hbar}} \left( \lambda + i \frac{\hat{P}}{mw} \right)
\]

\[
a' = \sqrt{\frac{mw}{2\hbar}} \left( \lambda - i \frac{\hat{P}}{mw} \right)
\]

7. (a) Consider 1-D harmonic oscillator problem and apply variational method to estimate the ground state energy.

Trial wave function \( \psi_0(x, \alpha) = A e^{-\alpha x^2} \)

Given \( \int x^2 e^{-\alpha x^2} \, dx = \frac{\pi^{\frac{3}{4}}}{\sqrt{\alpha}} 1.35 \ldots \frac{(2n+1)}{(2n)!} \).

(b) Given the information of ground state how do we obtain an upper bound to the 1st excited state using variational method.